

Chapter 3 - Characteristics of Structures

3.1 Introduction	2
3.1.1 Background.....	2
3.2 Basic Statical Determinacy	5
3.2.1 Introduction.....	5
3.2.2 Plane Beams and Frames	6
3.2.3 Plane Trusses	15
3.3 Stability.....	20
3.3.1 Introduction.....	20
3.3.2 Exceptions to Basic Rule.....	21
3.3.3 Examples.....	23
3.4 Further Statical Determinacy	25
3.4.1 Internal and External Determinacy.....	25
3.4.2 Cutting Back	28
3.4.3 Examples.....	30
3.5 Kinematic Determinacy	37
3.5.1 Definition.....	37
3.5.2 Examples.....	39
3.6 Problems.....	52

Rev. 1

3.1 Introduction

3.1.1 Background

Before analyzing a structure for the load effects, some questions must be answered:

Can the structure carry load?

This question is not about how much load the structure can carry, it is about if the structure can carry any load at all: is the structure stable? For example, we can imagine and model a pen standing on its point, and we could even posit that it could carry a load of (say) books in this position but it should be obvious that this theoretical model has two problems: it is very difficult (if not impossible) to balance a pen in this position, never mind load it with books, but even if this was achieved then the slightest force with horizontal component would cause catastrophic failure. Such a ‘structure’ is clearly inherently unstable and not suited to carrying people, furniture, machinery etc. We need to be able to identify structures with such instabilities before starting any kind of (utterly pointless) analysis. Only if a structure is stable do we then proceed to ask the next question.

What form of analysis is most suitable?

We have already seen that any analysis of a structure makes use of one or more of the pillars: equilibrium, compatibility of displacement, and constitutive relations. For some structures we can make use of the equations of static equilibrium alone: these are termed *statically determinate structures*, since the equations of static equilibrium alone are sufficient to ‘determine’ the responses to load. Other structures require the use of all three pillars and these are *statically indeterminate structures*, since the equations of static equilibrium alone are not sufficient to determine the responses to load.

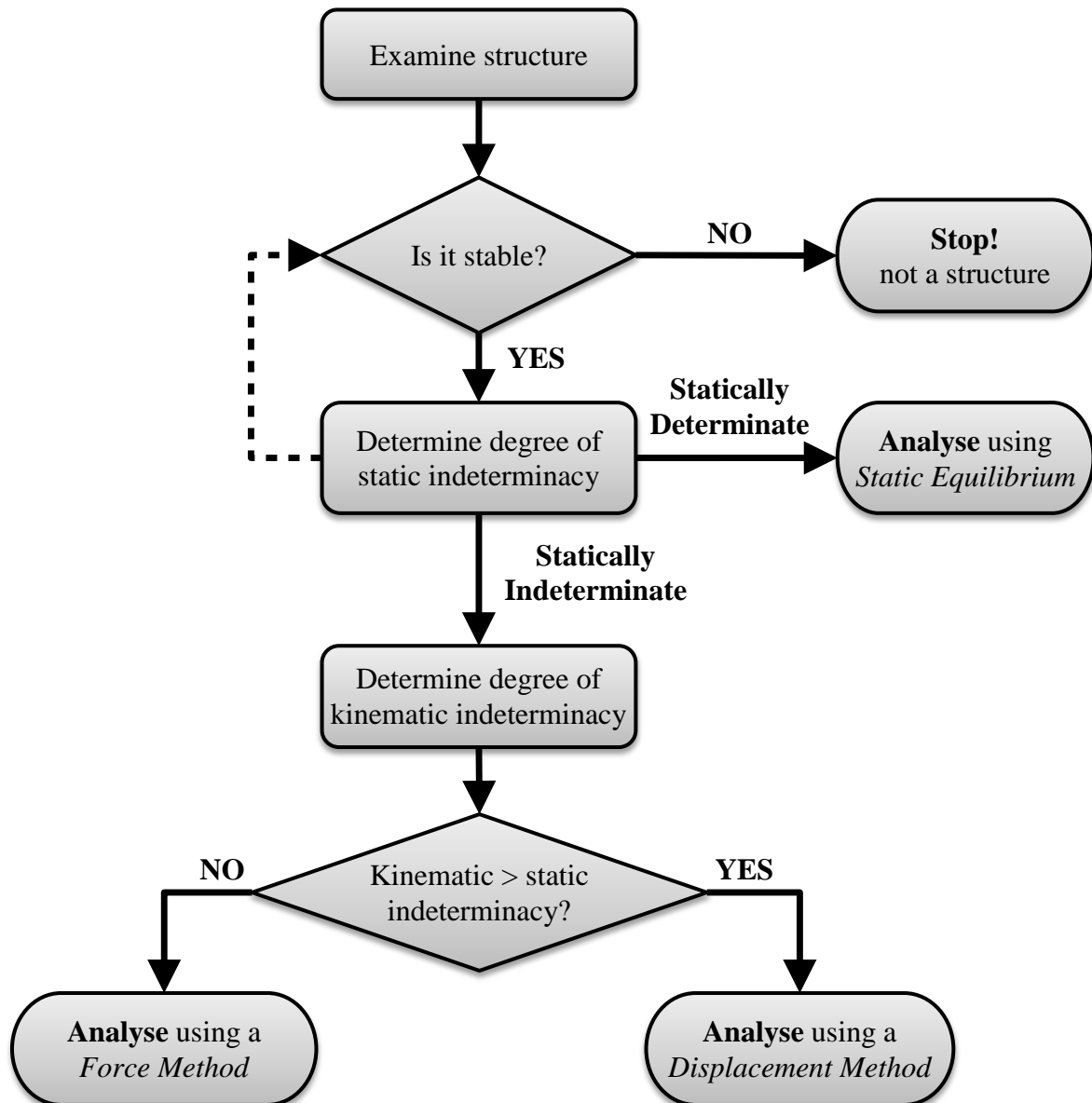
If a structure is statically indeterminate then a further decision is required before any analysis: whether the analysis will use a *force method* (in which single or multiple forces are the primary unknown), or a *displacement method* (in which single or multiple displacements are the primary unknown). It makes sense to choose the method that has the smaller number of unknowns that must be calculated. Since we already know the number of unknown forces (*degree of static indeterminacy*), we proceed to find the *degree of kinematic indeterminacy*, that is, the number of unknown independent joint displacements. We then choose our analysis method depending on which of these is smaller.

Summary

The flow chart shown below can help to visualize these steps: over time and with practice these steps will become second nature and intuitively obvious.

It should be noted that other factors can come into the decision to analyse using a force or displacement method. When the number of unknowns (static or kinematic) is nearly the same, the analyst might choose their preferred method, but when the number is very different the choice is more clear-cut. A significant exception is that almost all computer structural analysis packages use a form of the displacement method (a matrix stiffness or finite element method) regardless of the number of unknowns.

Finally, we will actually start our study with statical determinacy since it can also help to inform us of the stability of the structure.



3.2 Basic Statical Determinacy

3.2.1 Introduction

Statical determinacy of a structure refers to our ability to *determine* all reactions and load effects in the structure using *static* equilibrium alone. If we can, it is statically determinate, if we can't it is statically indeterminate.

From mathematics, we know that if we have the same number of equations as the number of unknown variables, then we can determine the solutions (i.e. the values of the unknowns). So, for *determinacy* of a structure, we need:

$$\text{No. unknowns} = \text{No. knowns}$$

If we have more unknowns than knowns, the structure is *indeterminate*:

$$\text{No. unknowns} > \text{No. knowns}$$

The amount to which it is statically indeterminate is the **degree of indeterminacy**, denoted n_s , and is thus given by:

$n_s = \text{No. unknowns} - \text{No. knowns}$

So we can also say:

$$n_s = 0 \rightarrow \text{Statically determinate}$$

$$n_s > 0 \rightarrow \text{Statically indeterminate}$$

3.2.2 Plane Beams and Frames

For a plane structure we have 3 known equations of equilibrium ($\sum F_x = 0$, $\sum F_y = 0$, $\sum M = 0$). If R is the number of reactions the structure can generate, then we have:

$R < 3$	Unstable
$R = 3$	Statically Determinate
$R > 3$	Statically Indeterminate

We will consider stability in a later section. For some plane structures we have other things we know, in addition to the 3 equations of statics. (For example, we know that at a hinge the bending moment is zero.) We term these *conditions* of the structure and the number of conditions, C , represents extra knowns and so the above equations become:

$R < 3 + C$	Unstable
$R = 3 + C$	Statically Determinate
$R > 3 + C$	Statically Indeterminate

And these are sometimes written:

$R - C < 3$	Unstable
$R - C = 3$	Statically Determinate
$R - C > 3$	Statically Indeterminate

These equations are valid in general, for plane beams, frames, and overall stability consideration of trusses (as will be explained later).

Finally then, we can determine the degree of indeterminacy as follows:

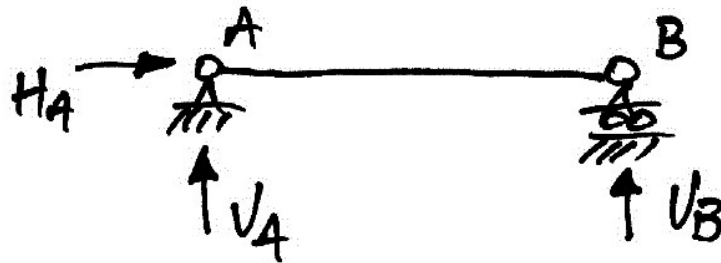
$$\begin{aligned}n_s &= \text{No. unknowns} - \text{No. knowns} \\ &= (R) - (3 + C)\end{aligned}$$

Giving the important equation for the **degree of static indeterminacy** of a plane frame:

$n_s = R - C - 3$

Examples

Example 1 – Simply-supported beam

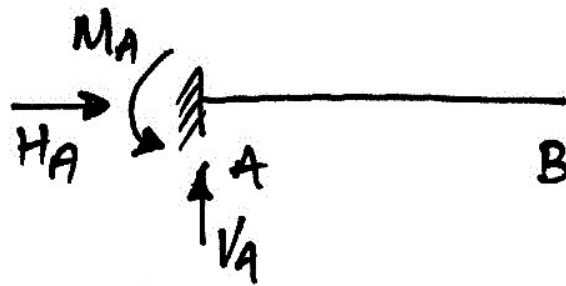


No. reactions, $R = 3$ No. conditions, $C = 0$

$$\begin{aligned} n_s &= R - C - 3 \\ &= 3 - 0 - 3 \\ &= 0 \end{aligned}$$

Therefore this structure is statically determinate.

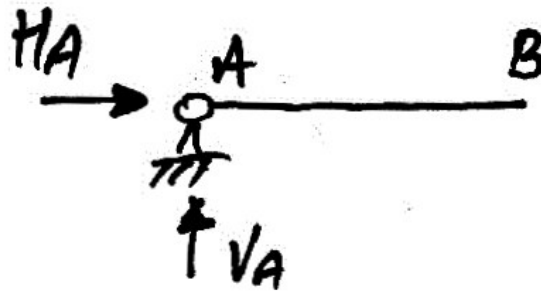
Example 2 – Cantilever



No. reactions, $R = 3$ No. conditions, $C = 0$

$$\begin{aligned} n_s &= R - C - 3 \\ &= 3 - 0 - 3 \\ &= 0 \end{aligned}$$

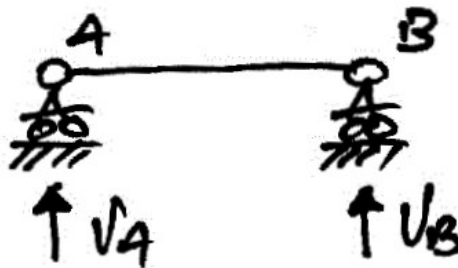
This structure is also statically determinate.

Example 3

No. reactions, $R = 2$ No. conditions, $C = 0$

$$\begin{aligned} n_s &= R - C - 3 \\ &= 2 - 0 - 3 \\ &= -1 \end{aligned}$$

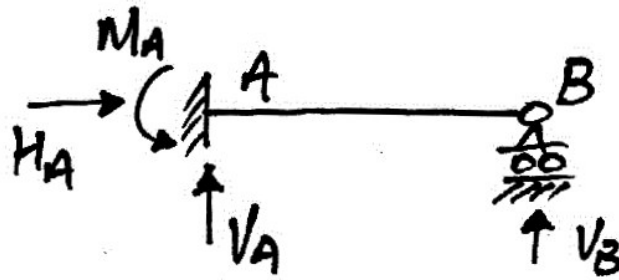
Therefore this structure is unstable: it cannot generate enough forces to resist one of the means in which of plane forces act (in this case it cannot resist rotation).

Example 4

No. reactions, $R = 2$ No. conditions, $C = 0$

$$\begin{aligned} n_s &= R - C - 3 \\ &= 2 - 0 - 3 \\ &= -1 \end{aligned}$$

This structure is also unstable since it cannot resist horizontal forces.

Example 5 – Propped CantileverNo. reactions, $R = 4$ No. conditions, $C = 0$

$$\begin{aligned}
 n_s &= R - C - 3 \\
 &= 4 - 0 - 3 \\
 &= 1
 \end{aligned}$$

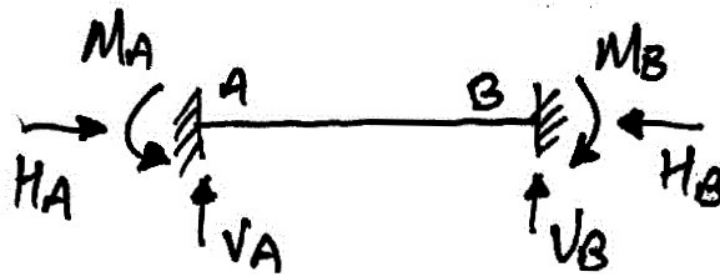
Therefore this structure is statically indeterminate to 1 degree.

Cutting back

In Example 5, notice that if the roller support at B was removed, then the structure reduces to a cantilever (Example 2) which is statically determinate. Since the roller offers one restraint, it took the removal of one restraint to get to a statically determinate structure, meaning that the degree of indeterminacy is 1. This is another way to arrive at the result termed ‘cutting back’. Similarly, we could remove the rotational restraint at A to arrive at the statically determinate simply support beam (Example 1). Thus the result of cutting back is independent of the restraints removed.

Further Examples

Example 6 – Fixed-Fixed beam

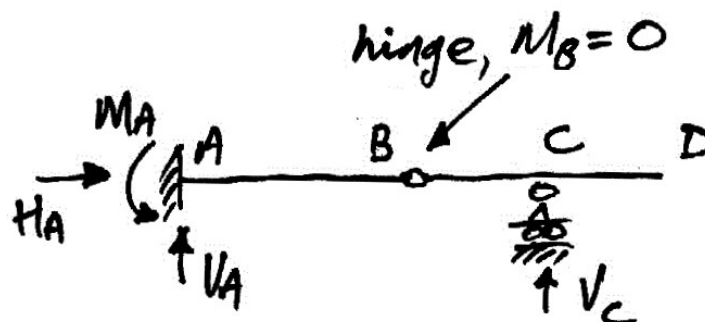


No. reactions, $R = 6$ No. conditions, $C = 0$

$$\begin{aligned} n_s &= R - C - 3 \\ &= 6 - 0 - 3 \\ &= 3 \end{aligned}$$

Therefore this structure is statically indeterminate to 3 degrees. Again notice that if the fixed support at B was removed then we have a statically determinate structure, the cantilever of Example 2. Since the fixed support offers three restraint, the degree of indeterminacy must be 3 for this structure.

Example 7

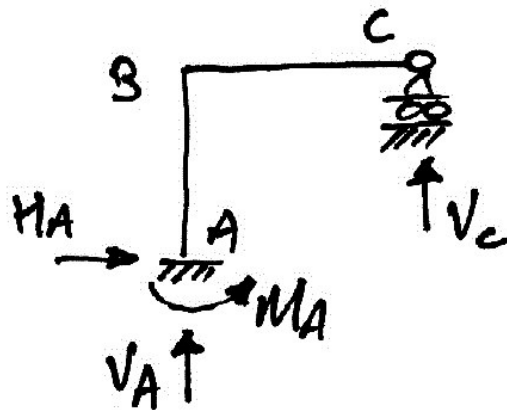


No. reactions, $R = 4$ No. conditions, $C = 1$

$$\begin{aligned} n_s &= R - C - 3 \\ &= 4 - 1 - 3 \\ &= 0 \end{aligned}$$

Therefore this structure is statically determinate. Also notice that without the hinge it is essentially a propped cantilever (Example 5) and so by inserting a hinge into a one-degree indeterminate structure we have arrived at a determinate structure. This is another form of cutting back: the removal of the restraint in this case is the removal of the ability to take moment at some point B along the beam. Further, if the hinge was to be inserted at A , this is the same as removing the rotational restraint at A (i.e. making it a pinned support) again resulting in a statically determinate structure (close to the simply-supported beam of Example 1).

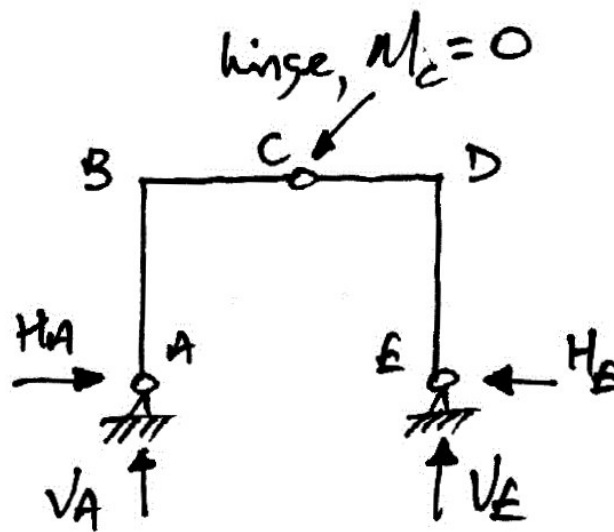
Example 8



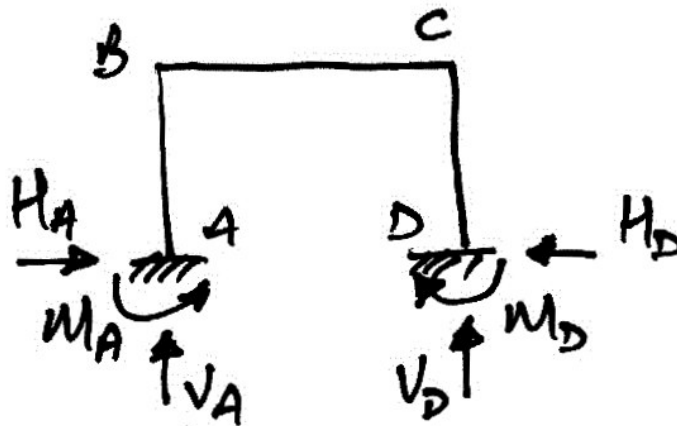
No. reactions, $R = 4$ No. conditions, $C = 0$

$$\begin{aligned} n_s &= R - C - 3 \\ &= 4 - 0 - 3 \\ &= 1 \end{aligned}$$

Therefore this structure is statically indeterminate to 1 degree. Following Example 7, we can see this is true by imaging a hinge inserted somewhere, perhaps at B , but if at A it is the same as making A a pinned support.

Example 9

$$\begin{aligned}
 n_s &= R - C - 3 \\
 &= 4 - 1 - 3 \\
 &= 0 \\
 &\rightarrow \text{Statically determinate}
 \end{aligned}$$

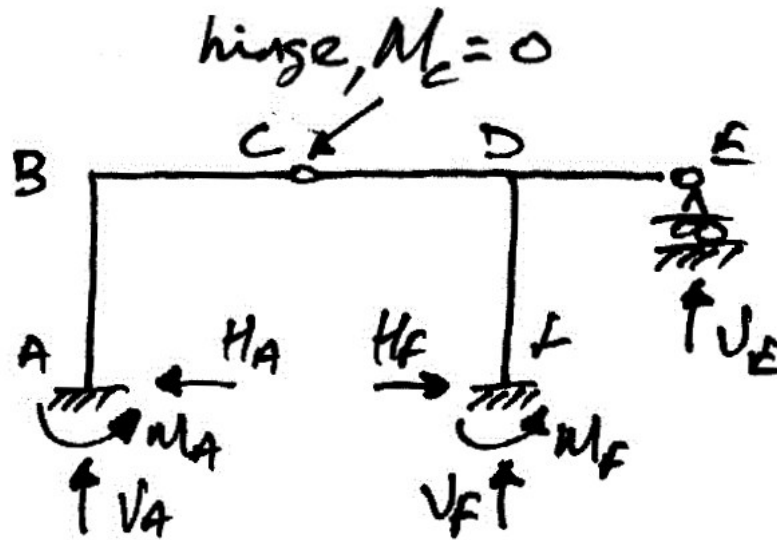
Example 10

$$\begin{aligned}
 n_s &= R - C - 3 \\
 &= 6 - 0 - 3 \\
 &= 3
 \end{aligned}$$

Therefore this structure is statically indeterminate to 3 degrees. If the support at D was removed the structure is essentially a cantilever (statically determinate) and so it

takes the removal of 3 restraints to render it determinate and so it is 3 degrees indeterminate.

Example 11



$$n_s = R - C - 3$$

$$= 7 - 1 - 3$$

$$= 3$$

→ 3 degrees statically indeterminate

3.2.3 Plane Trusses

For plane trusses, we must think carefully about the knowns and unknowns:

- Unknowns:

All of the reactions R , and the force in each member B are unknown;

- Knowns:

At each joint (or node) in a plane truss, we have 2 equations of statics that apply ($\sum F_x = 0, \sum F_y = 0$). Hence, with N joints (or nodes), we thus have $2N$ knowns.

Therefore, for a plane truss, we have:

$R + B < 2N$	Unstable
$R + B = 2N$	Statically Determinate
$R + B > 2N$	Statically Indeterminate

And so the **degree of static indeterminacy** is given by:

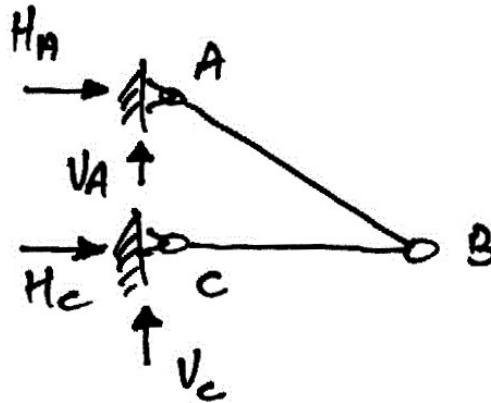
$$\begin{aligned}
 n_s &= \text{No. unknowns} - \text{No. knowns} \\
 &= (R + B) - (2N)
 \end{aligned}$$

Or:

$$n_s = R + B - 2N$$

Examples

Example 1



No. reactions, $R = 4$

No. members, $B = 2$

No. joints, $N = 3$

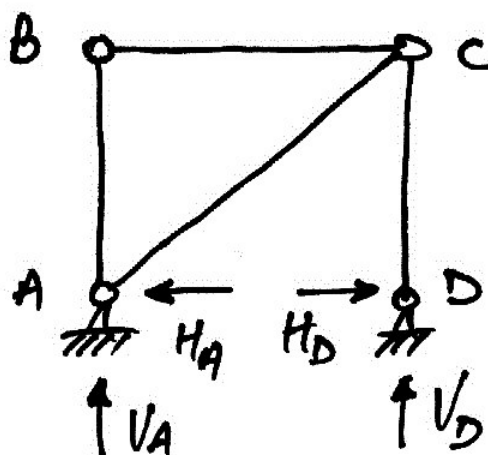
$$n_s = R + B - 2N$$

$$= 4 + 2 - 2(3)$$

$$= 0$$

And so this truss is a statically determinate truss.

Example 2

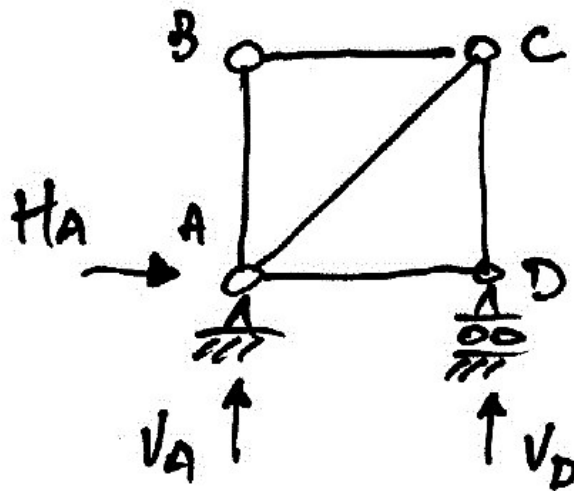


No. reactions, $R = 4$

No. members, $B = 4$

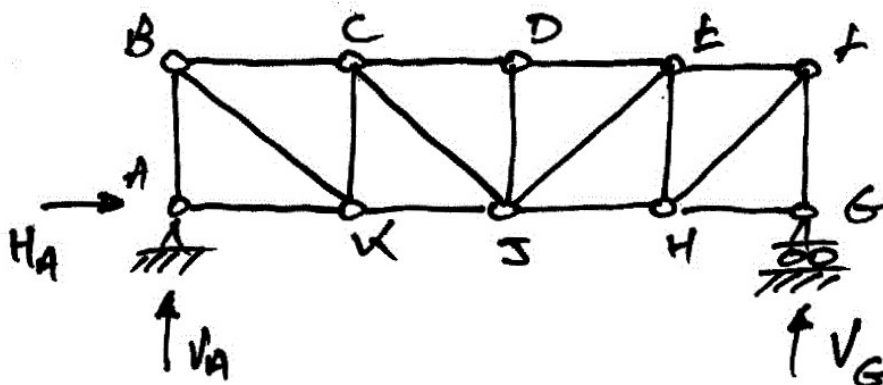
No. joints, $N = 4$

$$\begin{aligned}
 n_s &= R + B - 2N \\
 &= 4 + 4 - 2(4) \\
 &= 0 \\
 &\rightarrow \text{Statically determinate}
 \end{aligned}$$

Example 3

No. reactions, $R = 3$ No. members, $B = 5$ No. joints, $N = 4$

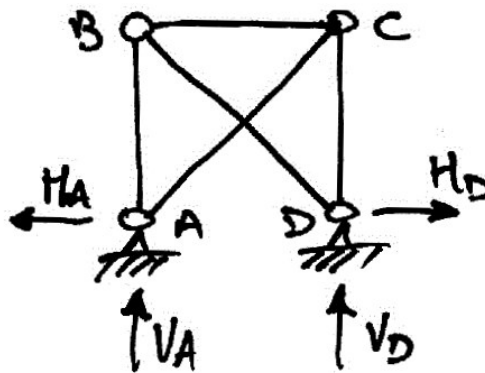
$$\begin{aligned}
 n_s &= R + B - 2N \\
 &= 3 + 5 - 2(4) \\
 &= 0 \\
 &\rightarrow \text{Statically determinate}
 \end{aligned}$$

Example 4

No. reactions, $R = 3$ No. members, $B = 17$ No. joints, $N = 10$

$$\begin{aligned}
 n_s &= R + B - 2N \\
 &= 3 + 17 - 2(10) \\
 &= 0 \\
 &\rightarrow \text{Statically determinate}
 \end{aligned}$$

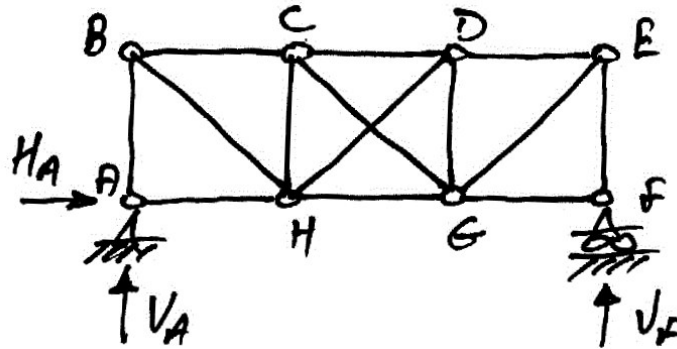
Example 5



No. reactions, $R = 4$ No. members, $B = 5$ No. joints, $N = 4$

$$\begin{aligned}
 n_s &= R + B - 2N \\
 &= 4 + 5 - 2(4) \\
 &= 1
 \end{aligned}$$

This the structure is 1 degree statically indeterminate. The addition of the extra member BD to the truss of Example 2 resulted in the truss becoming 1 degree statically indeterminate. This extra unknown is the axial force in the member. Further, reversing this process and removing the member BD from this truss results in a determinate truss (Example 2) and so this is again 1 degree indeterminate.

Example 6No. reactions, $R = 3$ No. members, $B = 14$ No. joints, $N = 8$

$$\begin{aligned}n_s &= R + B - 2N \\&= 3 + 14 - 2(8) \\&= 1\end{aligned}$$

So this truss is statically indeterminate to 1 degree.

3.3 Stability

3.3.1 Introduction

Structural stability is essential to ensure for real structures, since loads may come on at an angle even slightly different from those assumed in design. As designers of real structures, we must ensure this is possible. For example, we can draw a free-body diagram of our ‘theoretical pen’ showing that it can be stood on its tip but it’s a lot harder to do this in practice. The pen standing on its tip is not a stable structure!

Stability depends on the degree of statical indeterminacy, n_s , as follows:

- $n_s < 0$:

the structure is unstable as it does not have enough reactions to resist loads in each of the equations of equilibrium;

- $n_s = 0$:

A statically determinate structure: generally stable, unless it meets one of the exceptions;

- $n_s > 0$:

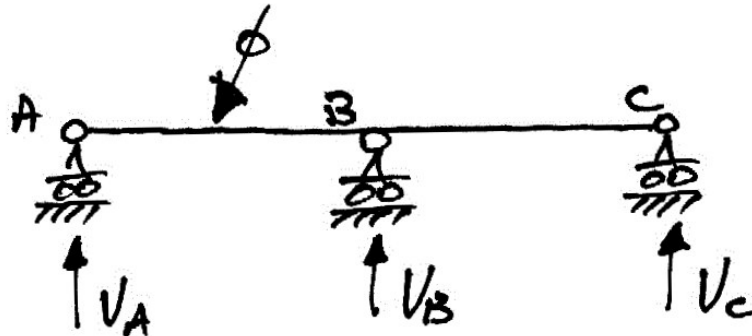
A statically indeterminate structure: generally stable, unless it meets one of the exceptions.

So, once a structure has sufficient number of reactions to be statically determinate or indeterminate, it is, in general, stable unless an exception exists (to be explained). Note that if a structure is unstable, it is irrelevant to report its determinacy: if it is unstable it cannot carry load and so strictly is not a structure! Finally, a structure may also be **partially unstable** where only part of the structure is unstable. This can be due to one of the exceptions applying to part of the structure. These exceptions are explained next.

3.3.2 Exceptions to Basic Rule

Exception 1: Reactions are Parallel

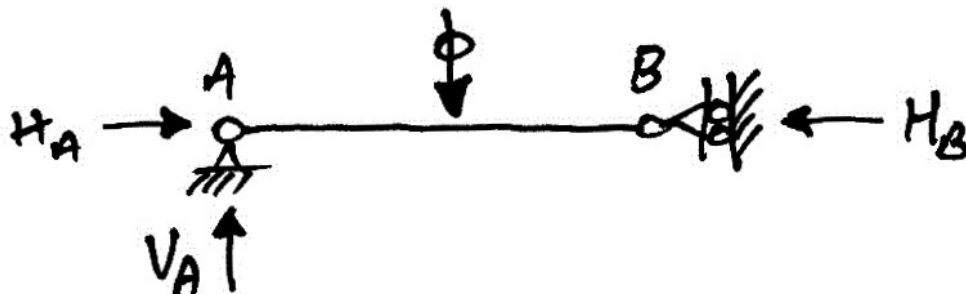
If all of the reactions are parallel (even if the structure is indeterminate), then it cannot resist loads in another direction.



The above structure is not stable, even though the number of reactions is 3 and may thus have been thought as statically determinate.

Exception 2: Reactions are Concurrent

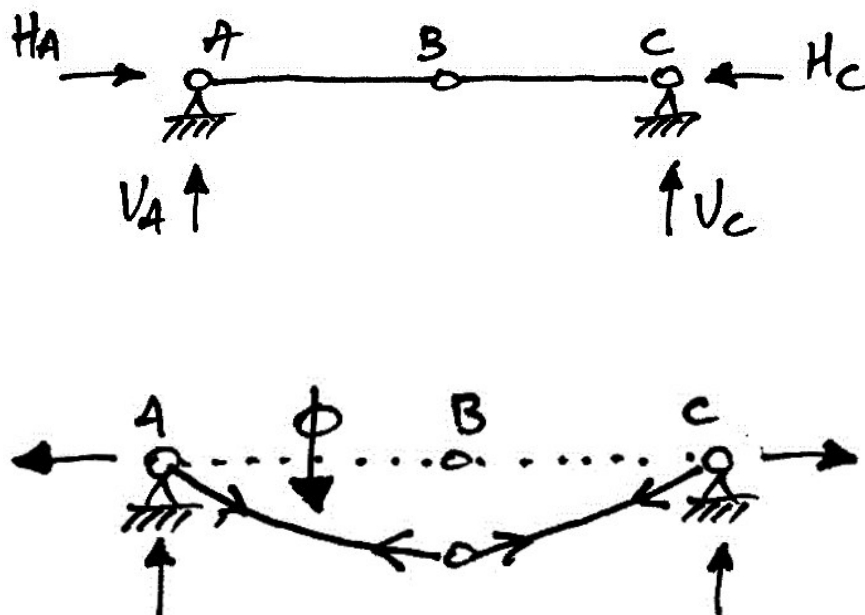
If the lines of action of all reactions pass through the same point, then the structure will rotate about that point and is thus unstable.



This structure again has three reactions, but is not stable, since it will rotate about point A, the point where the lines of actions of all reactions pass through.

Exception 3: Unserviceable Structures

There are situations when it is not clear whether or not a structure can sustain its loads without undue deformation. It may need to move from its initial geometric configuration into another before it can sustain the loads. For real structures, this movement to another configuration is unacceptable when the structure is to be in service and so this situation can be called unstable.



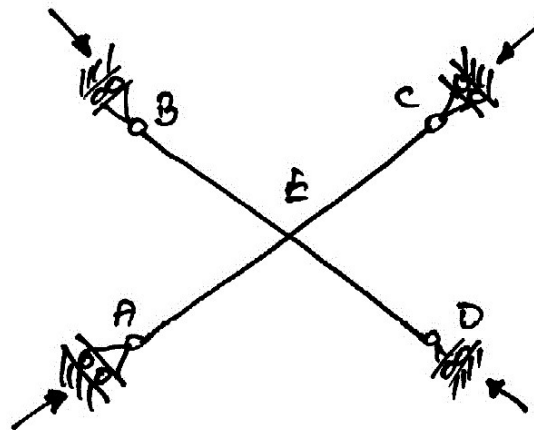
This structure deflects from its initial configuration until the members go into tension thus resisting the load as a kind of truss. However, since this shape is very different to its starting shape (i.e. members may now be at 10°), the structure is unstable due to not being suitable for service.

There are situations when as structural engineers we want a structure to carry loads in an unserviceable configuration, typically after the removal of a column to ensure robustness of structures.

3.3.3 Examples

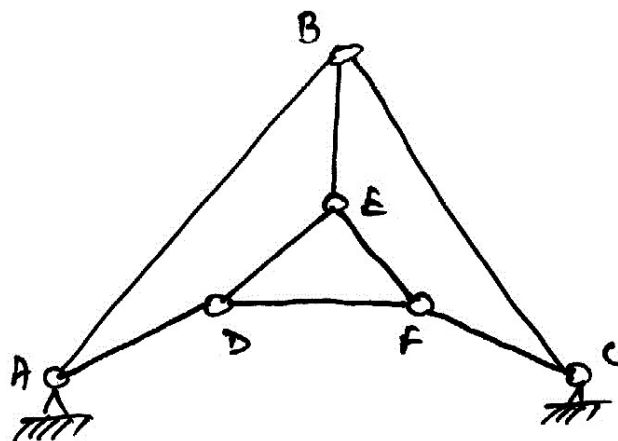
Example 1

This structure has four reactions, but their lines of action all pass through the centre point. It is thus unstable. It could however, be a mechanical model of a propeller.



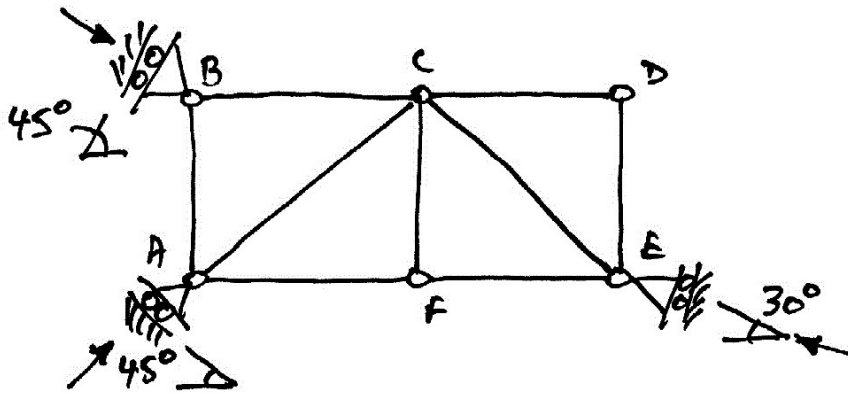
Example 2

The outermost triangle is a stable structure. However, a load in the central triangle of this structure would cause the internal members to rotate until a suitable geometric configuration in which the loads could be carried was arrived at. Thus as presented (in its initial configuration) the structure is partially unstable.

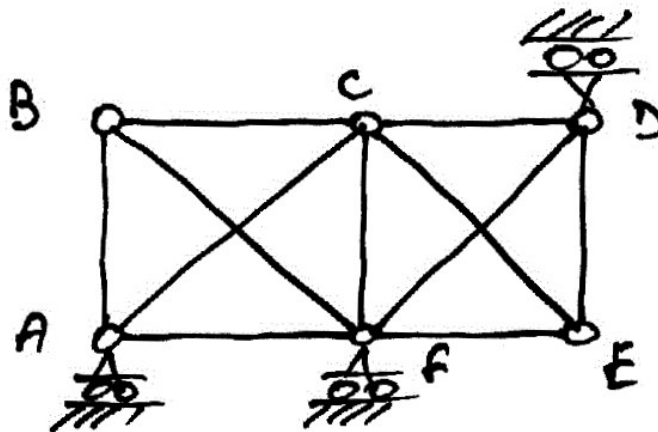


Example 3

The lines of actions of the three reactions all pass through the centre point of the first panel and the structure will thus rotate about this point. It is therefore unstable.

**Example 4**

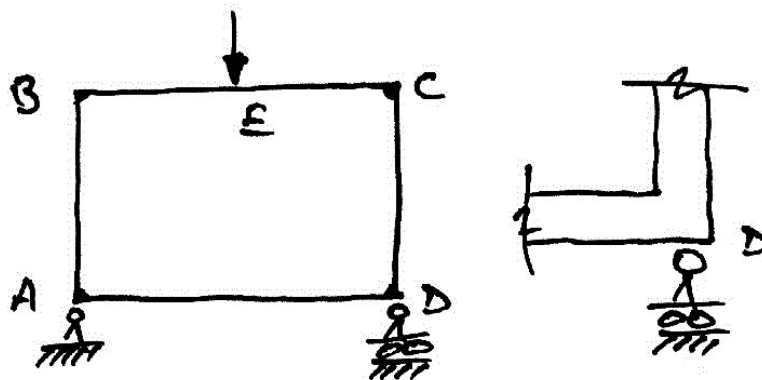
This structure is unstable as all reactions are parallel.



3.4 Further Statical Determinacy

3.4.1 Internal and External Determinacy

Consider the structure shown below. It is representative of the structural model for a culvert or a single storey of a sway-frame building perhaps.



Applying our previous formulation for static determinacy of this structure reveals:

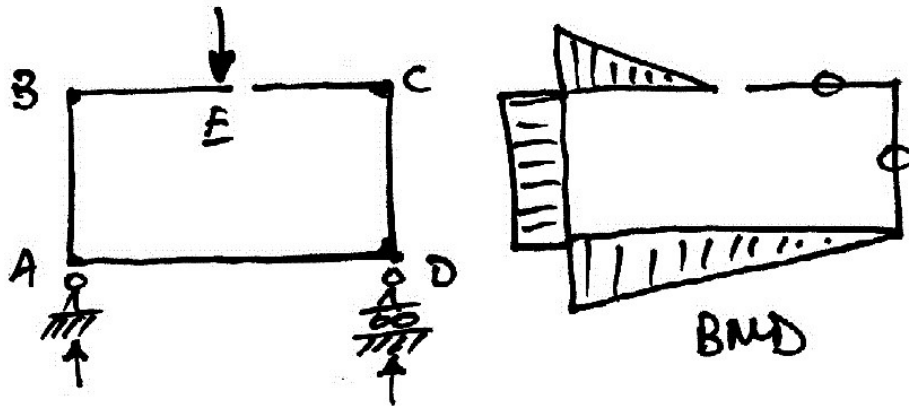
$$n_s = R - C - 3 = 3 - 0 - 3 = 0$$

And so this structure appears to be statically determinate. However, because of the continuity of the frame ABCD members at the joints (see the example joint *D* inset), it is not possible to determine the bending moments or other internal stress resultants. Only the reactions can be determined using the equations alone. Since these reactions are external forces to the structure, we therefore conclude that this structure is:

- *externally* statically determinate, and;
- *internally* statically indeterminate.

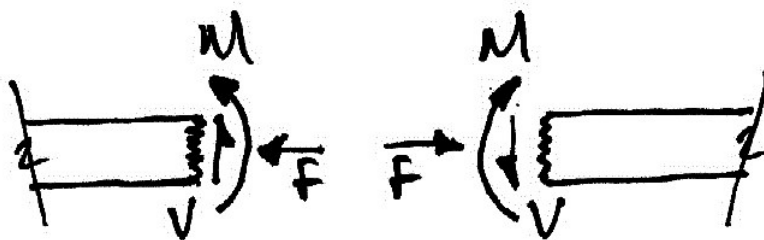
Therefore, *overall*, it is indeterminate. We had to distinguish between the external and internal forces and stress resultants to come to this conclusion.

To see that this is indeed the case for this example structure, if we imagine a cut at E in the structure, the bending moments (and other load effects) can be found as shown below:



Now the structure is statically determinate – the equations of static equilibrium are sufficient to solve for all load effects (bending moment, shear etc.). We therefore ask; what did the cut change about the frame so that it is now determinate?

For a flexural member, a cut releases three internal stress results at the point, bending moment, shear force, and axial force, as shown below:



Since it took the removal of three unknowns to make the structure determinate, in its original state it is therefore 3 degree statically indeterminate, and these indeterminacies are internal.

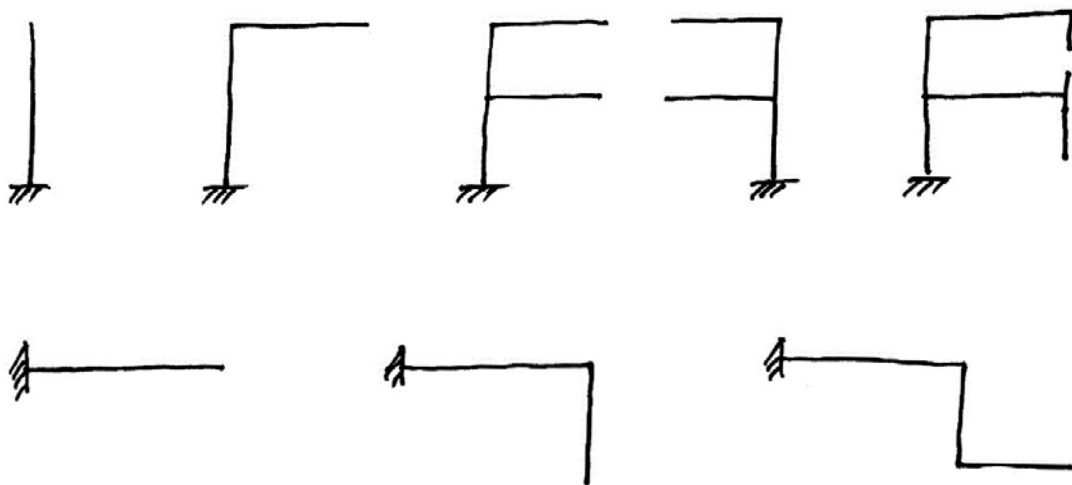
Any structure that has a complete 'ring' of members will be internally indeterminate to some degree, depending on the number of such rings. Cutting back is a good strategy for identifying the degree of statical indeterminacy for these structures.

3.4.2 Cutting Back

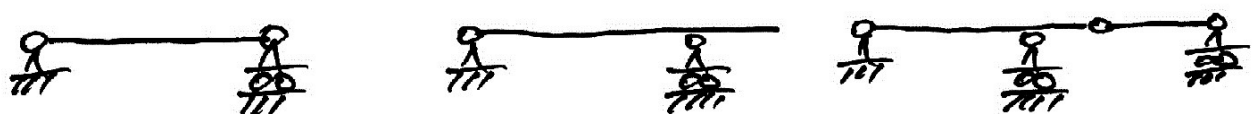
In cutting back we do not only seek to remove restraints, sometime it is useful to add restraints initially. The total amount of restraints added and removed then informs the degree of statical indeterminacy. In doing this though, we are seeking base structural forms, about which we have knowledge of the statical determinacy.

Base Structural Forms

A 'base' structural form in this context is one about which we have knowledge of the statical determinacy: we try to reveal one of these forms in cutting back the structure. For example, a simple cantilever can be extended to arrive at closer relationships to other forms of structure, as shown:



All of these are statically determinate. Similarly, the simply supported beam is a useful base structure to identify in a more complex one:



Removing Releases – Adding Restraints

Finally, often it is useful to remove hinges by adding a moment restraint; making the problem worse initially, but facilitating identification of the base structures:

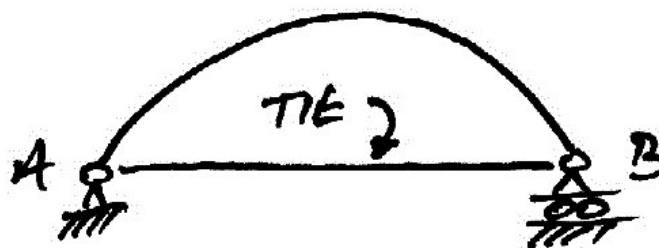


This concept can be extended further to supports and so on to facilitate arriving at base structures quickly.

Ties/Links/Struts

Some structures have elements identified through one of these terms which indicate that it carries axial force only. In such case, cutting this member releases one unknown.

As a basic example we can see that cutting the tie in the tied arch shown below renders a determinate system. Therefore, since one restraint was removed to render the determinate base structural form (simply-supported beam), the tied arch as shown is a 1 degree statically indeterminate structure.

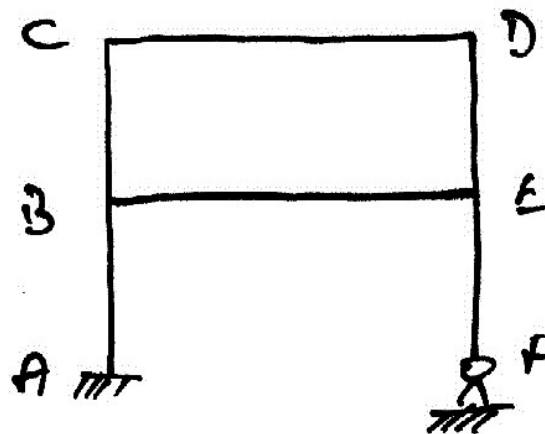


3.4.3 Examples

Example 1

Problem

Is the following structure stable? If stable, state the degree of statical indeterminacy.



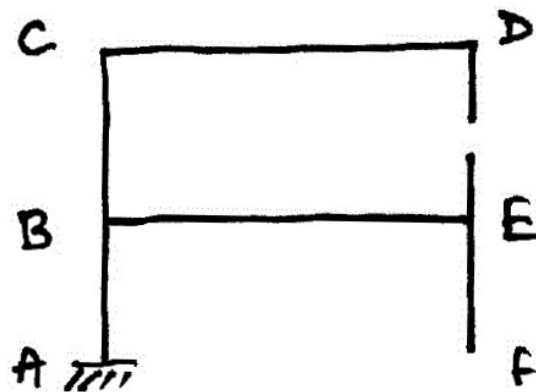
Solution

We notice that this structure has 5 reactions and no hinges (conditions) and so is ostensibly 2 degrees statically indeterminate. Therefore it is stable. However, we also note that $BCDE$ forms a ring and so has internal indeterminacies also.

At this point we ‘cut back’ this structure to try to get a base statically determinate structure that we are familiar with, as follows:

- Introduce a cut in member (say) CD to break the ring. In doing so we release 3 unknowns;
- Remove the support at F releasing two further unknowns.

After this we are left with an obvious statically determinate structure as shown:

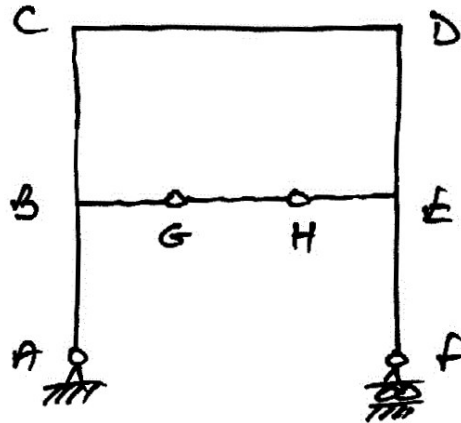


Hence we can summarize the original structure as follows:

- Externally statically indeterminate to 2 degrees;
- Internally indeterminate to 3 degrees;
- Overall it is $n_s = 5$ degrees statically indeterminate.

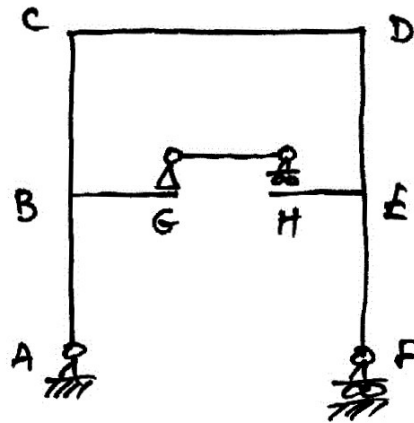
Example 2**Problem**

Is the following structure stable? If stable, state the degree of statical indeterminacy.

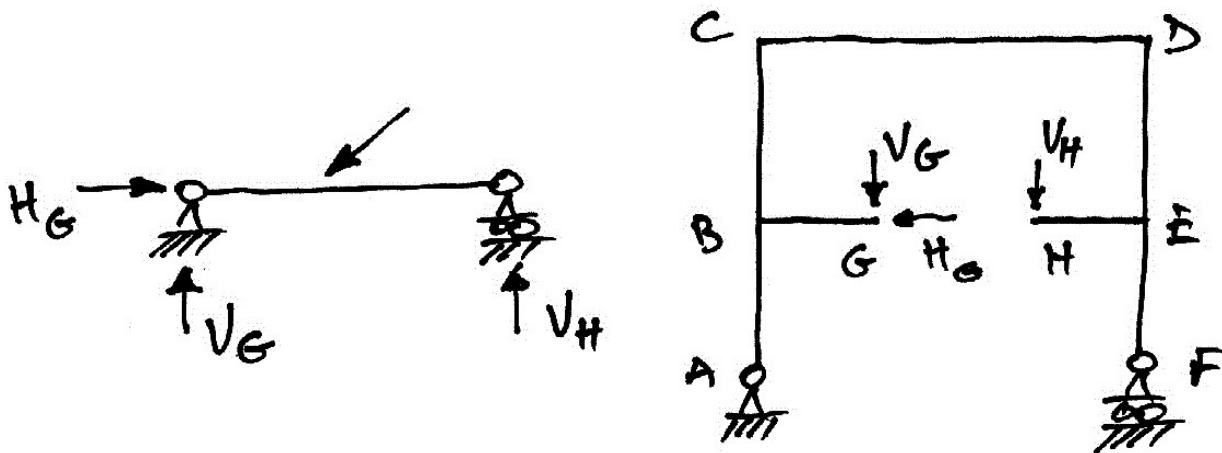
**Solution 1**

This structure appears statically determinate from the supports alone (i.e. externally statically determinate). However, the members $BCDEGH$ form a ring, even if not a complete ring due to the hinges G and H .

Members with hinges at both ends can only transfer axial force. To explain this, recall that a hinge resists two stress resultants, shear and axial force. By releasing the ability to carry axial force at (say) H , we effectively have the following structure:



Should any load be applied to the member GH it will transfer to the remaining structure as shown below:



The sequence is:

- Action: the load is the action applied to member GH ;
- Reaction: the 'supports' of member GH (offered by the rest of the structure) generate reactions to the load;
- Action: these reactions in turn apply themselves to the rest of the structure (the beam sits on the cantilevers).

This structure is clearly statically determinate. Since the release of one restraint resulted in a statically determinate structure, the original structure is 1 degree statically indeterminate.

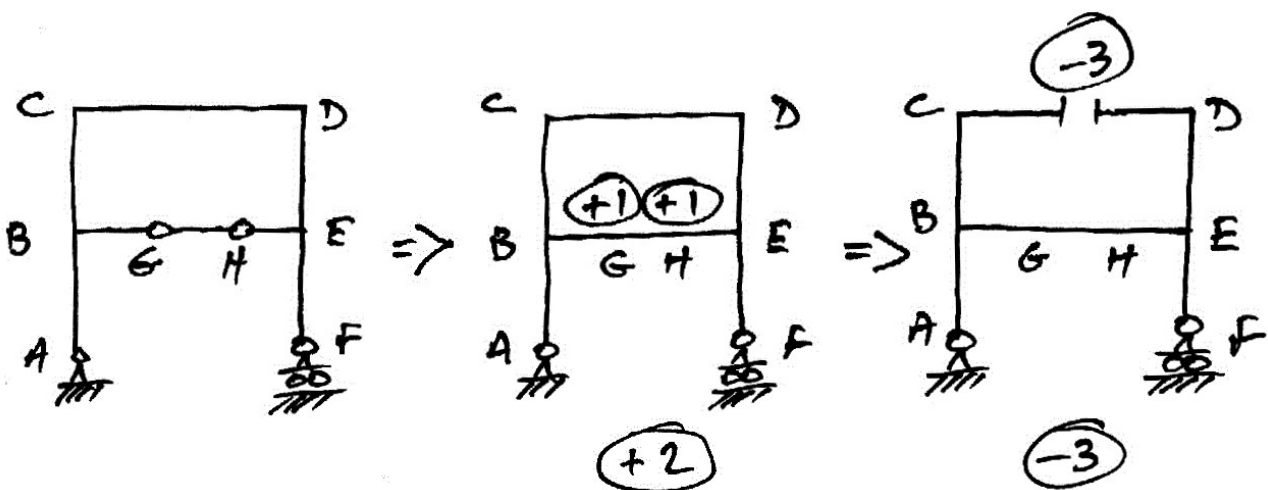
Solution 2

An alternative means of solving this problem is illustrated in the following diagrams and described as follows:

- first make the problem 'worse' by adding the ability to carry moment at the hinges, this results in a more usual 'ring' in the structure;
- this ring is then treated as normal: it is broken by releasing 3 unknowns;
- Since the remaining structure is statically determinate, we then have:

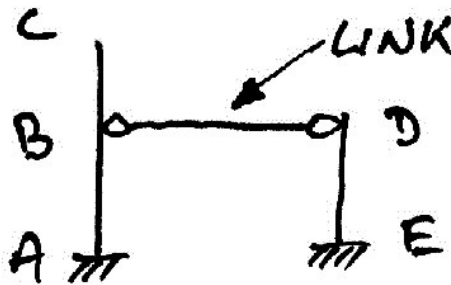
$$n_s = -(+2 - 3) = 1$$

Since the net change in unknowns is $(+2 - 3) = -1$ to get to a statically determinate structure, $n_s = 1$, which is the result previously arrived at.



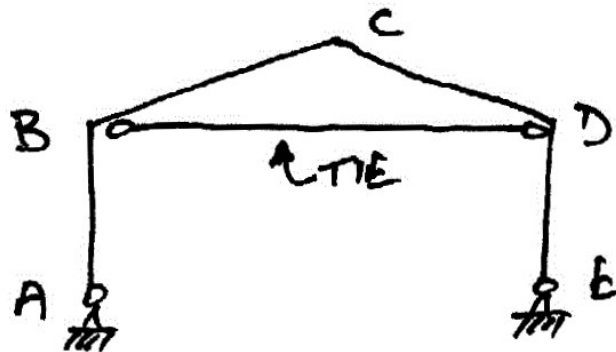
Further Examples

Example 3

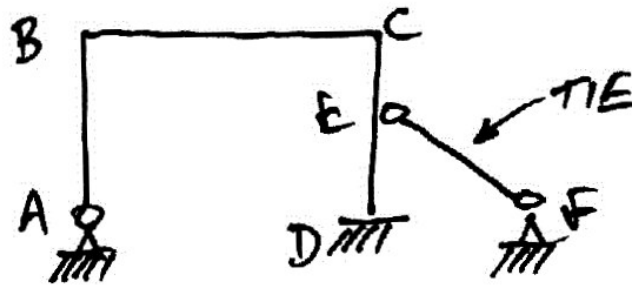


In this case cutting the link releases one unknown (its axial force) and renders two cantilevers (base structural forms). Hence the structure is one-degree statically indeterminate. This is in spite of the fact that there are 6 reactions.

Example 4



Again cutting the tie here releases one internal redundancy. A further release, say the horizontal reaction at *E* renders a base structural form (simply-supported beam/frame). Consequently the original structure is 2 degrees statically indeterminate.

Example 5

There are several alternative approaches to solving this structure, here are just two:

Approach 1:

- Cut the tie (-1 restraint);
- Make D a roller (-2 restraints);

A simply-supported frame results and so $n_s = -(-1 - 2) = 3$.

Approach 2:

- Cut the tie (-1 restraint);
- Make A a fixed support (+1 restraint);
- Cut member BC (-3 restraints).

Two cantilevers result and so $n_s = -(-1 + 1 - 3) = 3$ again as it should.

3.5 Kinematic Determinacy

3.5.1 Definition

Just as statical determinacy relates to knowledge about the unknown forces in a structure, kinematic determinacy relates to the knowledge we have about the movements of a structure. In particular, *kinematic indeterminacy*, n_k , is defined as:

“the number of unknown independent joint displacements”

Each part of this definition is considered:

- *Displacements:*
any possible rotation or translation of a joint. For plane frames, there are three per joint, x - and y -translations, and a rotation.
- *Joint:*
this is the end of any member;
- *Unknown:*
Displacements that are known, say at a support, are not included.
- *Independent:*
If a relationship is known between several displacements from the geometry of the frame, or an assumption about the behaviour of the structure, then these displacements count as one unknown displacement, since the others can be found once one is known. It is this aspect of the definition that requires most care.

It must be emphasized that there is no particular relationship between statical and kinematic determinacy: one does not infer the other.

Two assumptions about the behaviour of plane frames that are particularly relevant to the determination of kinematic indeterminacy are:

1. Only small displacements are considered and so second order effects are neglected.
2. It is assumed that flexural members (members that carry load primarily through bending) do not change length, that is, we neglected axial deformation (NAD).

The implication of these assumptions for kinematic indeterminacy is that relationships can be found between the movements of joints, and so they are not independent, affecting the result as described above. This arises as follows from each assumption respectively:

- joints move perpendicular to the member longitudinal axis;
- joints do not move parallel to the longitudinal axis.

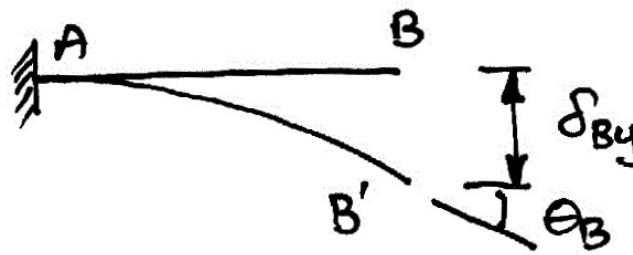
Lastly, two points that will arise later are:

- Any form of loading must be considered or allowed for generally – this is important in considering possible deflected shapes of the structure;
- It is not possible to have a relationship between rotations of different joints since a moment load may be put at one of the joints changing their relative values. Hence joint rotations are always independent.

3.5.2 Examples

Example 1 – Cantilever

The kinematic determinacy of the cantilever is best seen by imagining a possible displacement curve as follows:



Since joint A is fixed, there are no unknown displacements, whilst at B it is possible that there are 3 (δ_x , δ_y , θ) but since we neglect axial deformations joint B can only move vertically. There are no relationships between the displacements. Consequently there are only two unknown joint displacements as so the cantilever is kinematically indeterminate to 2 degrees:

$$n_k = 2$$

Note that this statically determinate structure is kinematically indeterminate. Finally, we will often summarize the calculations as follows:

Joint	δ_x	δ_y	θ
A	✗	✗	✗
B	✗ (NAD)	✓	✓

$n_k = 2$

Example 2 – Fixed-Fixed Beam

For this structure the table of the joints and their possible unknown displacements is:



Joint	δ_x	δ_y	θ
A	x	x	x
B	x	x	x

$n_k = 0$

And so this is a kinematically determinate structure (whilst being 3 degrees statically indeterminate).

Example 3 – Simply-Supported Beam

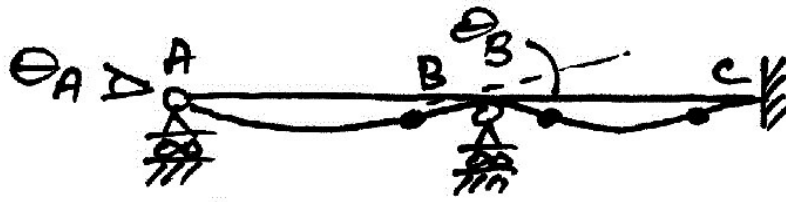
For this statically determinate structure we have:



Joint	δ_x	δ_y	θ
A	x	x	✓
B	x (NAD)	x	✓

$n_k = 2$

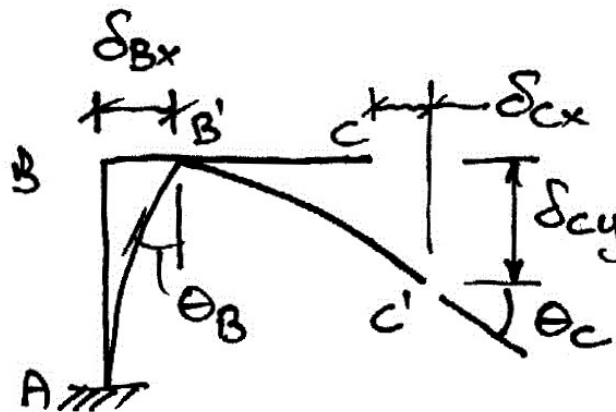
Note the importance of the assumption of neglecting axial deformation.

Example 4 – Continuous Beam

Joint	δ_x	δ_y	θ
A	x (NAD)	x	✓
B	x (NAD)	x	✓
C	x	x	x

$$n_k = 2$$

So this statically indeterminate structure has the same kinematic determinacy as a simply-supported beam. This is why a displacement method of analysis (such as moment distribution) is more suitable than a force method.

Example 5- Simple Frame

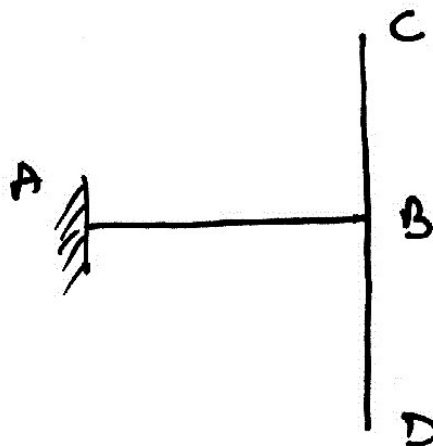
Joint	δ_x	δ_y	θ
A	x	x	x
B	✓ (Δ)	x (NAD)	✓
C	✓ (Δ)	✓	✓

$$n_k = 4$$

For this frame the small deflection and NAD assumptions mean that the horizontal displacement of joint C must be the same as that of joint B . This relationship is identified as Δ in the table. Since these displacements are not independent, they contribute just one kinematic unknown. The remaining unknowns arise from the joint rotations and vertical movement of joint C .

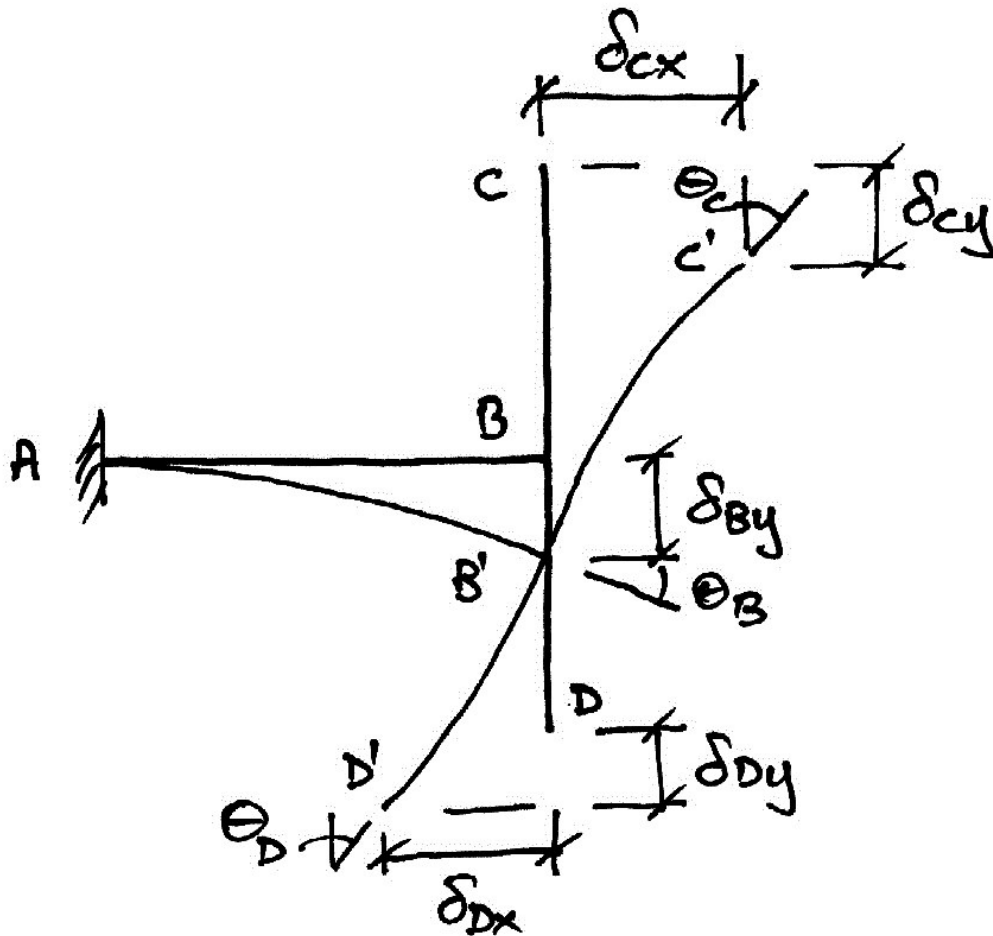
Example 6

The structure is shown below:



For this structure it is necessary to consider a possible general deflected shape that may occur. However, any form of loading that can be applied to this structure must be accounted for in the kinematic determinacy, so the deflected shape should be very general and not have in-built relationships that depend on a particular form of loading.

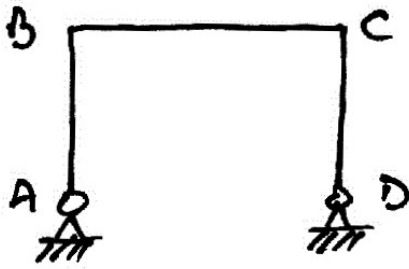
The possible deflected shape is:



Joint	δ_x	δ_y	θ
A	x	x	x
B	x (NAD)	✓ (Δ)	✓
C	✓	✓ (Δ)	✓
D	✓	✓ (Δ)	✓

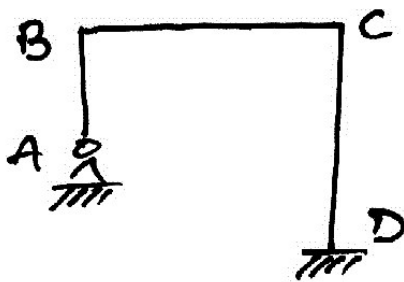
$$n_k = 6$$

For this structure, the overall number of possible joint displacements is 8, but since by the assumption of NAD, the y-displacements of joints B, C, and D must all be equal (since their connecting members do not change length), these three displacements are not independent and are related through some value Δ .

Example 7

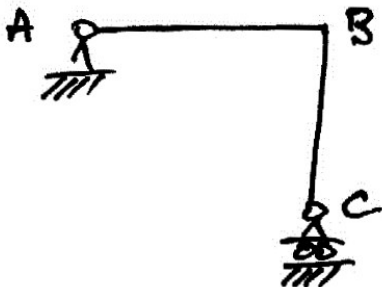
Joint	δ_x	δ_y	θ
A	x	x	✓
B	✓ (Δ)	x (NAD)	✓
C	✓ (Δ)	x (NAD)	✓
D	x	x	✓

$n_k = 6$

Example 8

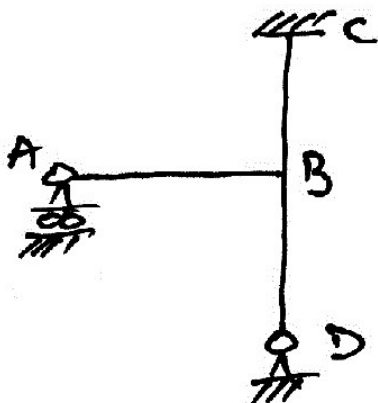
Joint	δ_x	δ_y	θ
A	x	x	✓
B	✓ (Δ)	x (NAD)	✓
C	✓ (Δ)	x (NAD)	✓
D	x	x	x

$n_k = 5$

Example 9

Joint	δ_x	δ_y	θ
A	x	x	✓
B	x (NAD)	x (NAD)	✓
C	✓	x	✓

$n_k = 4$

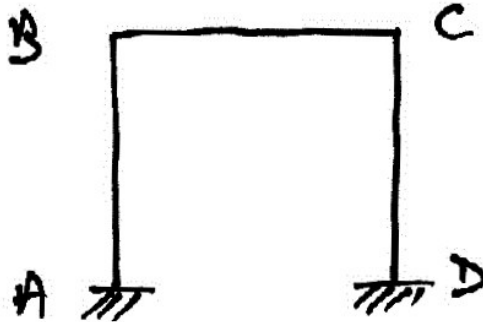
Example 8

Joint	δ_x	δ_y	θ
A	✓ (Δ)	x	✓
B	✓ (Δ)	x (NAD)	✓
C	x	x	x
D	x	x	✓

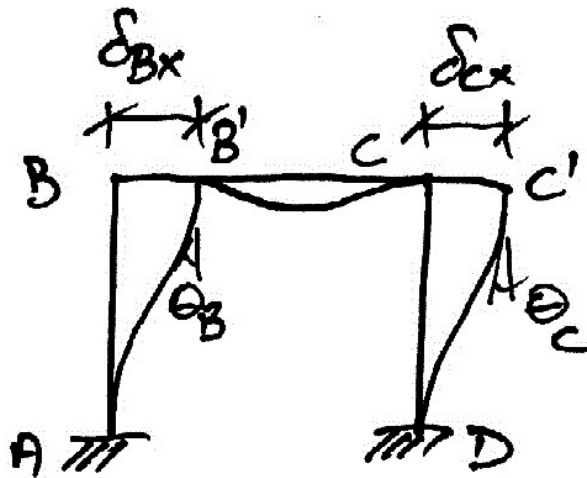
$n_k = 4$

Example 9

For this example we will consider some cases of particular load that can be useful to acknowledge in certain circumstances. The frame is as shown below:



And the most general deflected shape is shown with the table:



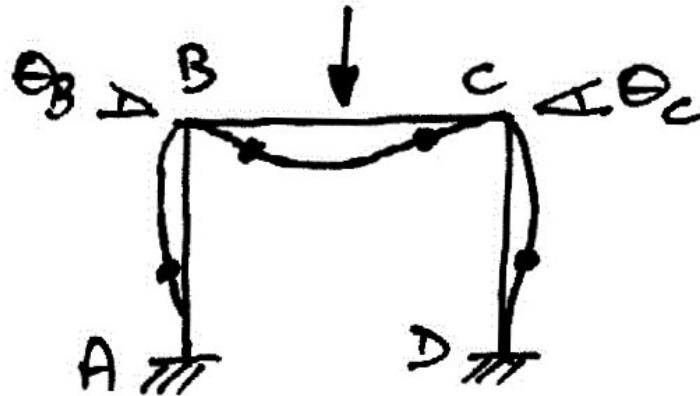
Joint	δ_x	δ_y	θ
A	x	x	x
B	✓ (Δ)	x (NAD)	✓
C	✓ (Δ)	x (NAD)	✓
D	x	x	x

$$n_k = 3$$

Considering only a point load in the middle of the beam, by symmetry we have:

- No sway (lateral movement) of the frame ($\delta_{Bx} = \delta_{Cx} = 0$);
- Symmetry of the rotations ($\theta_B = -\theta_C = \theta$).

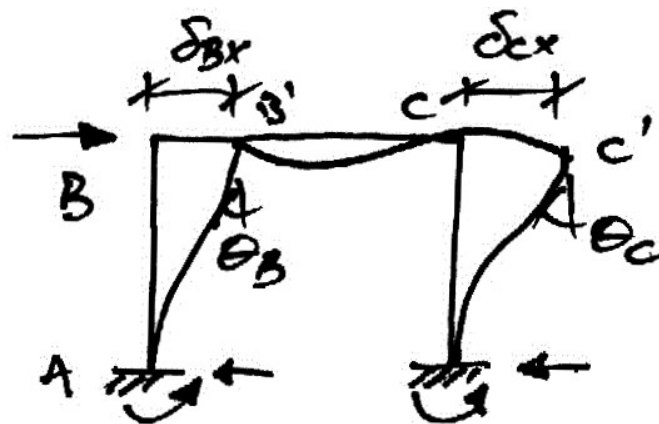
Thus in this special case we have $n_k = 1$.



For another special case, relevant to wind and earthquake loading, we take only a point load at B as shown to get:

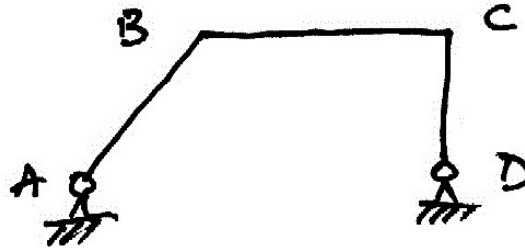
- Sway (lateral movement) of the frame ($\delta_{Bx} = \delta_{Cx} = \Delta$ say);
- Symmetry of the rotations ($\theta_B = \theta_C = \theta$ say).

And so the frame under this specific form of loading is $n_k = 2$.

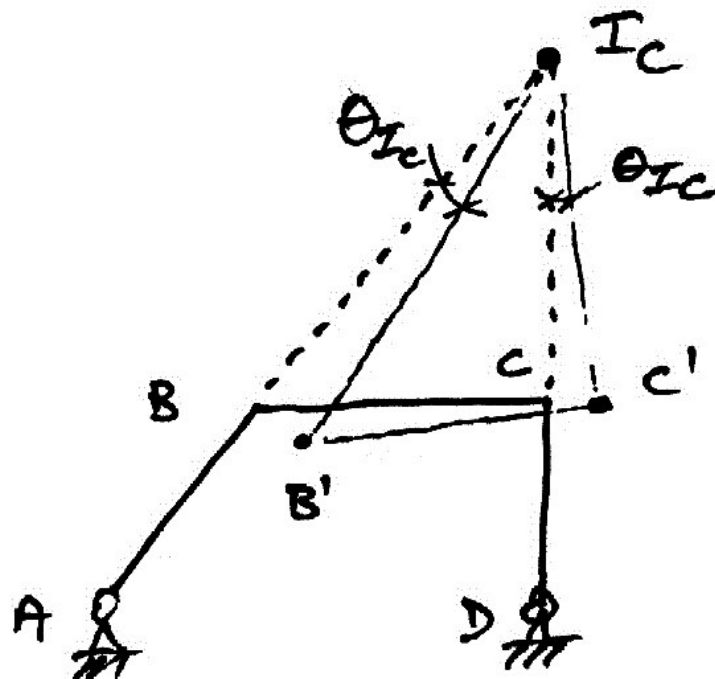


Example 10

For this example we consider a frame for which the concept of the *instantaneous centre of rotation*, I_c , is very useful.



Recall that projection of the lines of the members meet at the I_c , and that the lamina (triangle) thus formed rotates about I_c as a lamina (i.e. it does not alter shape) by an angle θ_{I_c} . Thus the movement of the joints can be seen in the diagram below from which it is apparent that the movements of joints B and C have some geometrical relationship to θ_{I_c} :



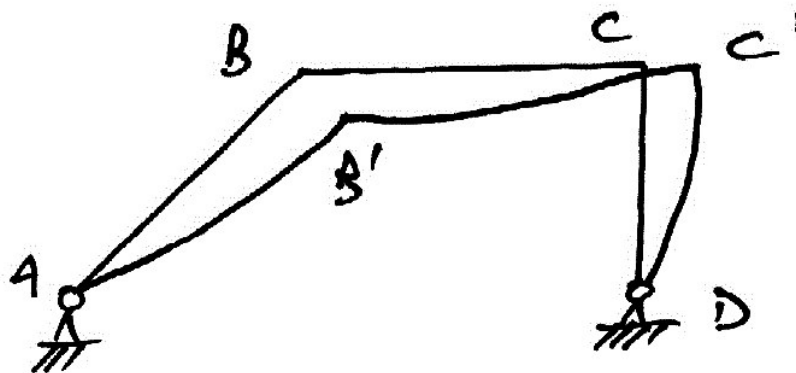
Therefore the table is:

Joint	δ_x	δ_y	θ
A	✗	✗	✓
B	✓ (θ_{I_c})	✗ (θ_{I_c})	✓
C	✓ (θ_{I_c})	✗ (NAD)	✓
D	✗	✗	✓

$n_k = 5$

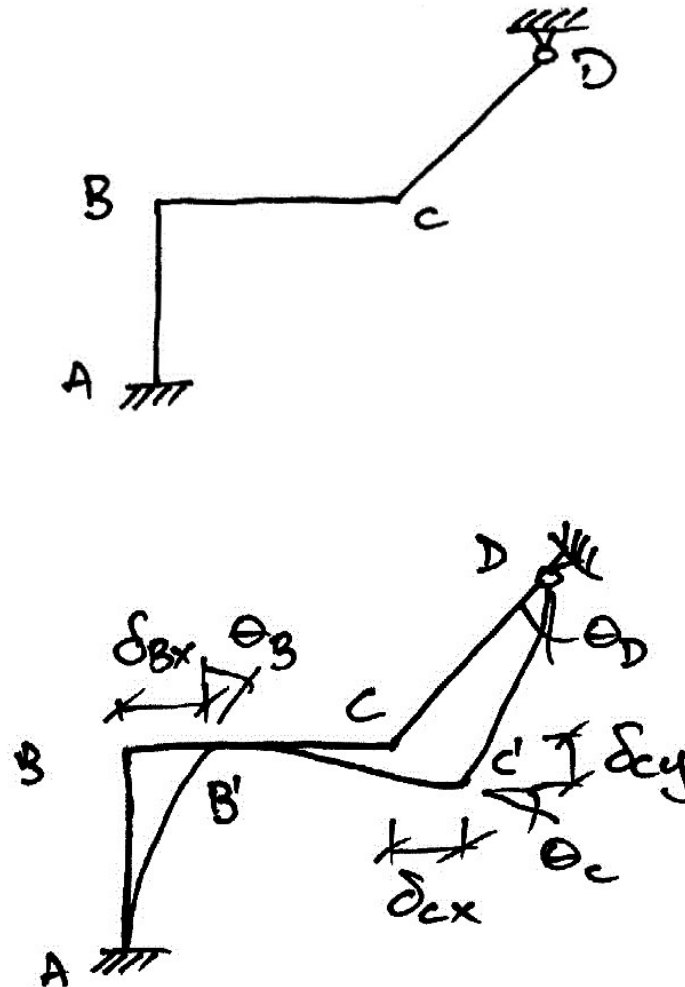
It is because of the two assumptions (small deflections and NAD) that complete knowledge of the displaced positions of joints B and C is known in relation to θ_{I_c} .

The actual deflected shape will look similar the following, under one form of loading:



Example 11

This structure is below, and a possible deflected shape and the table given after.



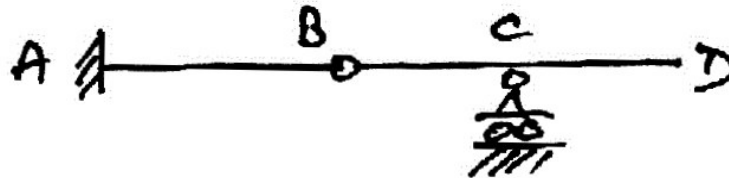
Projection of member CD and AB show that the frame moves about an IC somewhere below joint A . Hence we know we can relate the translations of joints B and C to this:

Joint	δ_x	δ_y	θ
A	x	x	x
B	✓ (θ_{I_c})	x (NAD)	✓
C	✓ (θ_{I_c})	✓ (θ_{I_c})	✓
D	x	x	✓

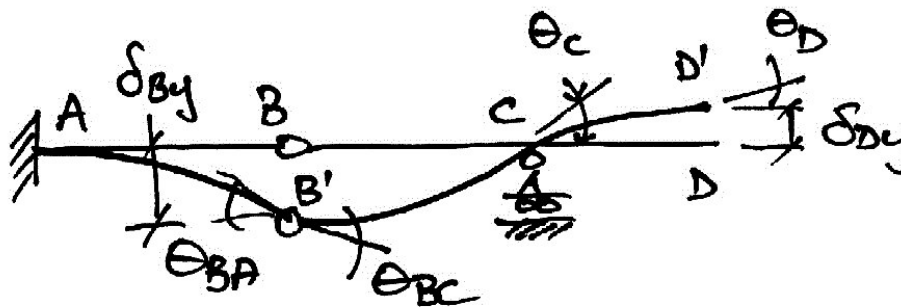
$n_k = 4$

Example 12 – Beam with Hinge

This structure is below, and again a possible deflected shape after.



The main difficulty with the hinge is that there are two rotations at B , due to the discontinuity in rotation (or slope) that the hinge causes.



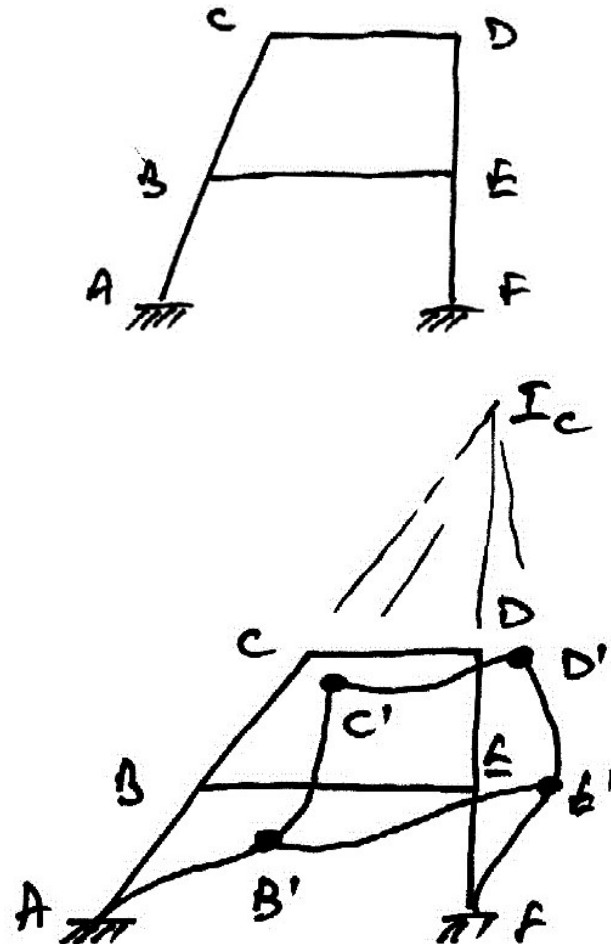
Hence we have:

Joint	δ_x	δ_y	θ
A	x	x	x
B	x (NAD)	✓	✓✓ (θ_{BA}, θ_{BC})
C	x (NAD)	x	✓
D	x (NAD)	✓	✓

$$n_k = 6$$

Example 13

For this structure the IC is very useful as shown below.



Giving

Joint	δ_x	δ_y	θ
A	x	x	x
B	✓ (θ_{I_c})	✓ (θ_{I_c})	✓
C	✓ (θ_{I_c})	✓ (θ_{I_c})	✓
D	x (NAD)	✓ (θ_{I_c})	✓
E	x (NAD)	✓ (θ_{I_c})	✓
F	x	x	x

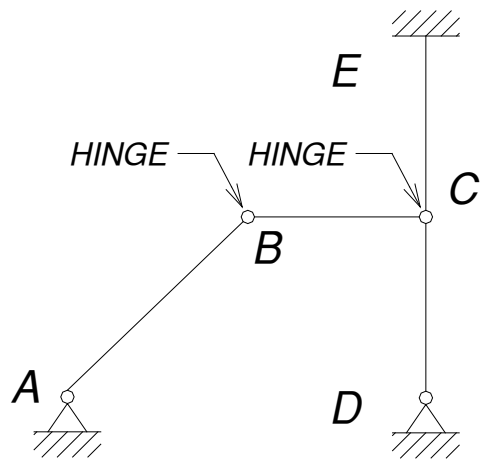
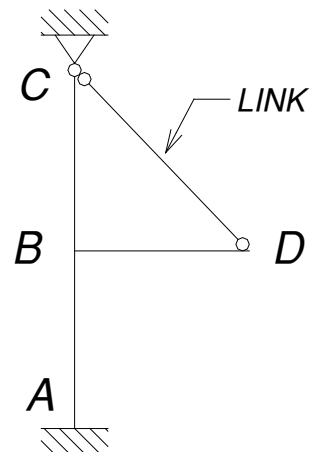
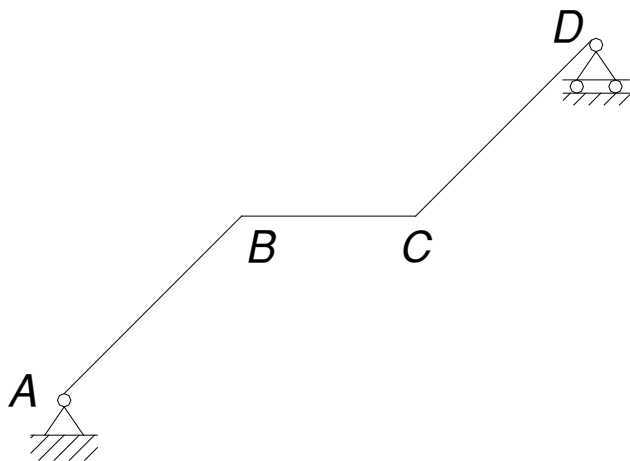
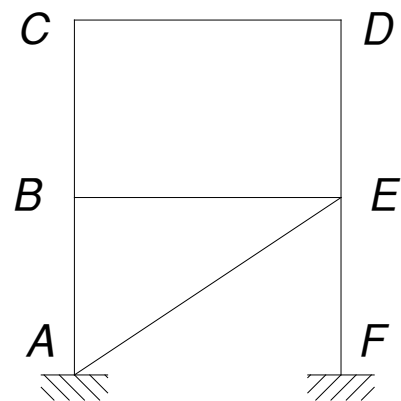
$$n_k = 5$$

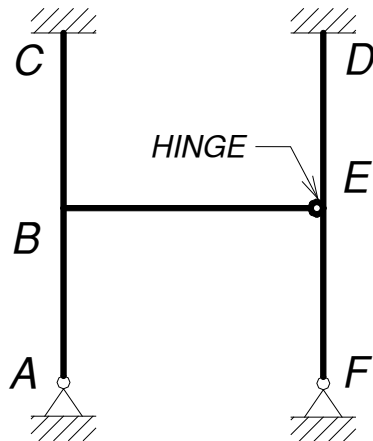
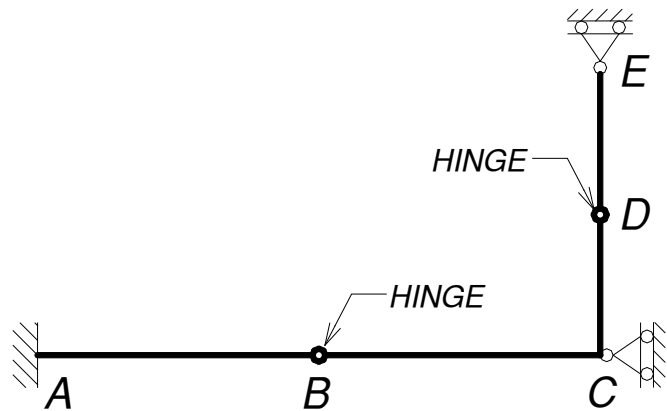
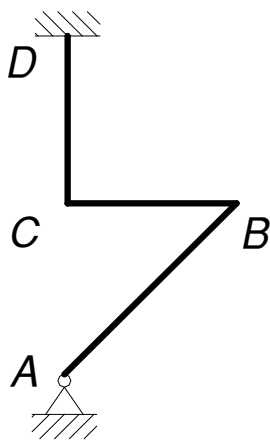
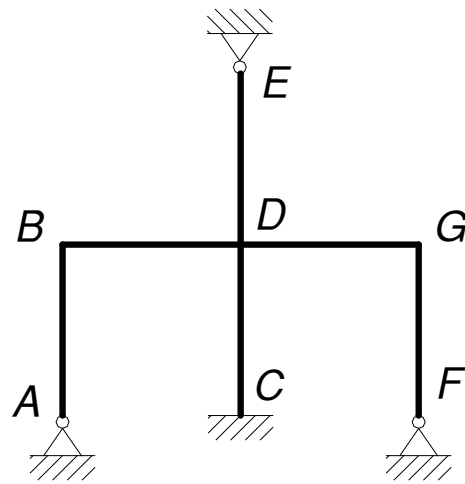
3.6 Problems

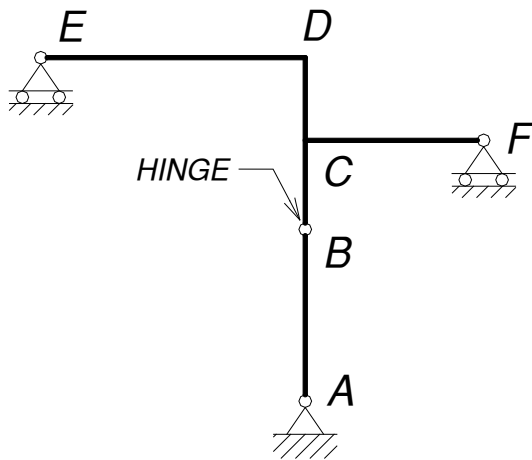
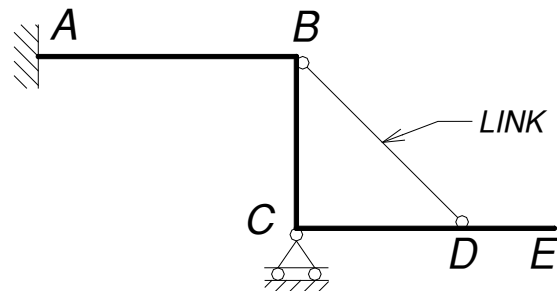
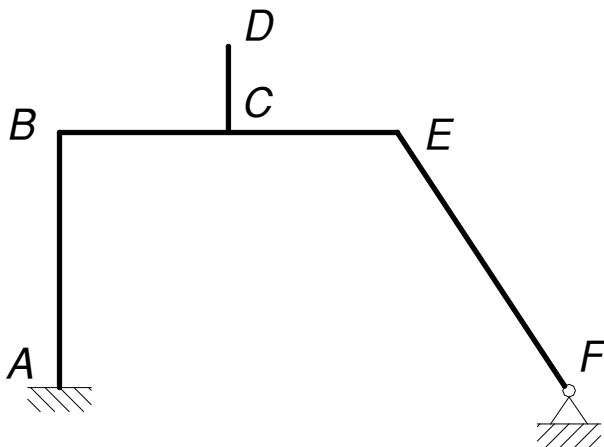
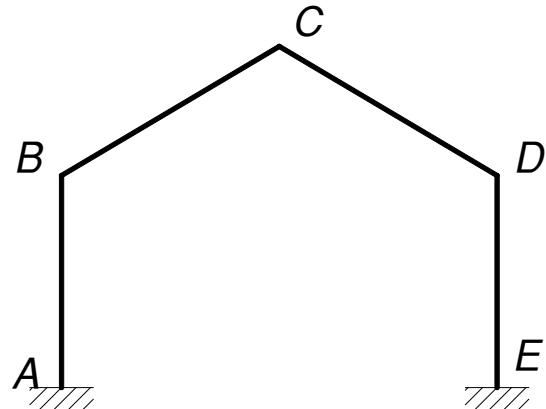
The following problems are taken from exams over previous years. For each problem:

- Classify each of the structures shown in Fig. Q1(a), indicating whether the structure is unstable or stable and statically determinate or indeterminate (giving the degree of indeterminacy).
- Determine the degree of kinematic indeterminacy of the structures shown in Fig. Q1(b), stating briefly reasons for your answer and any assumptions made about axial deformations.

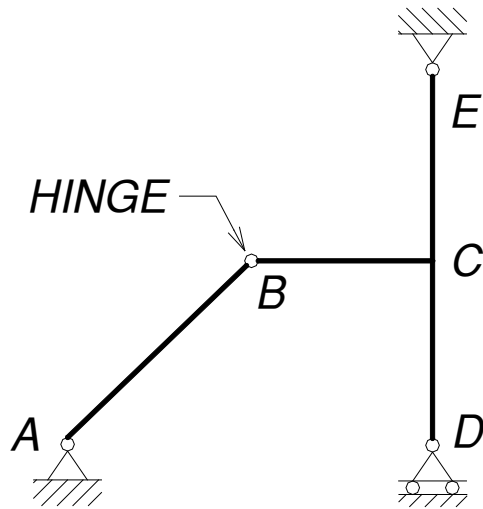
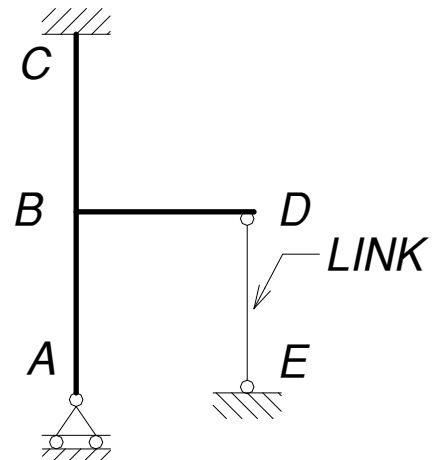
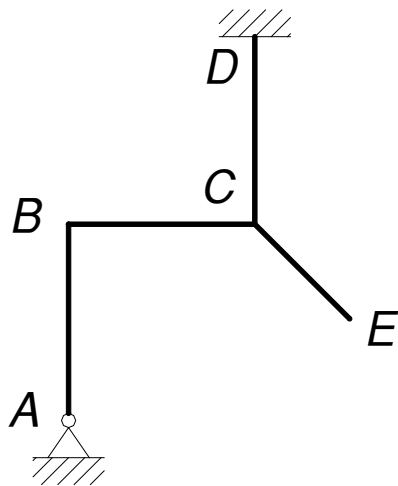
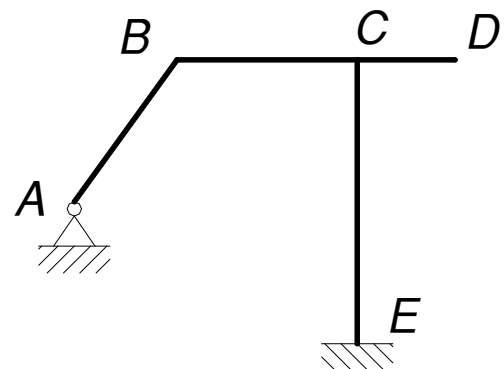
Summer, 2006/07

**FIG. Q1(a)(i)****FIG. Q1(a)(ii)****FIG. Q1(b)(i)****FIG. Q1(b)(ii)**

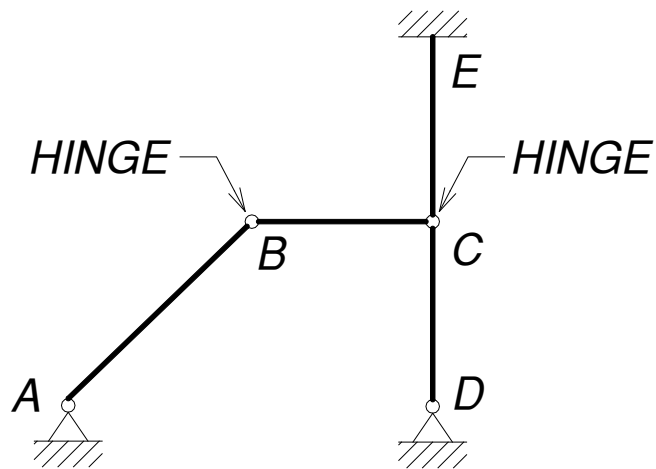
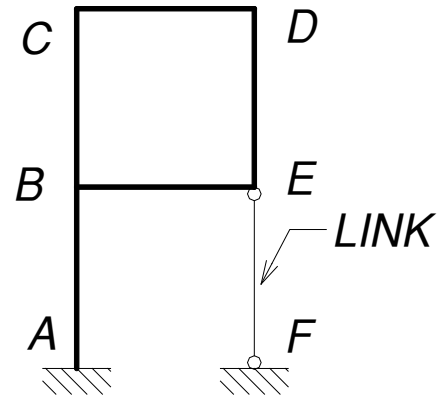
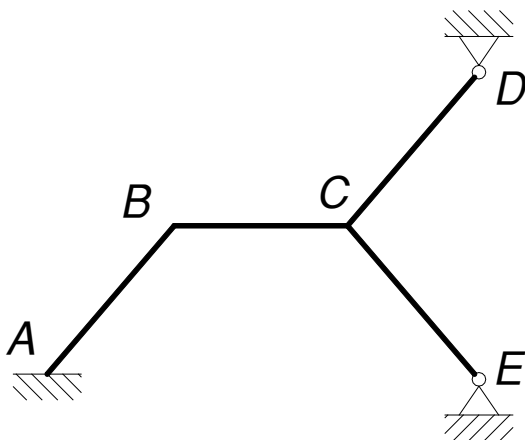
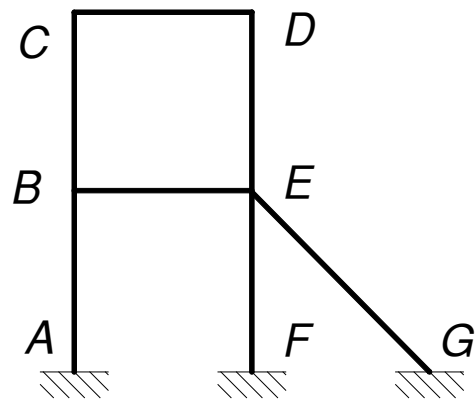
Semester 1, 2007/08**FIG. Q1(a)(i)****FIG. Q1(a)(ii)****FIG. Q1(b)(i)****FIG. Q1(b)(ii)**

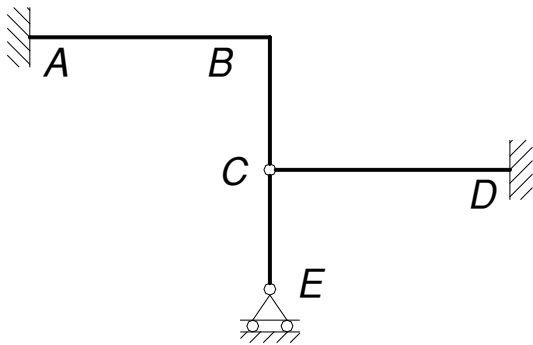
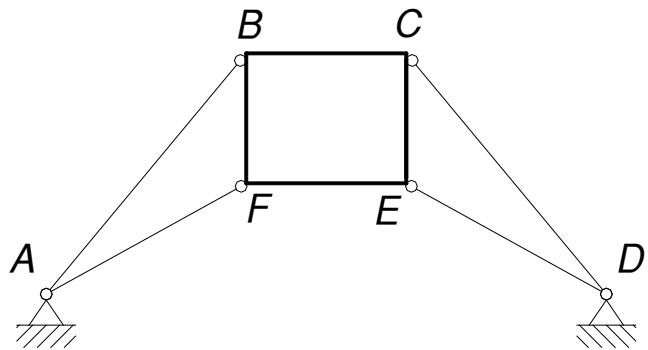
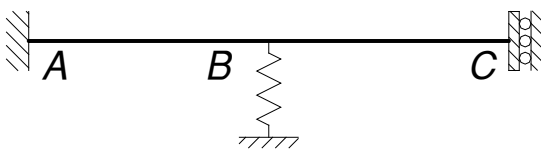
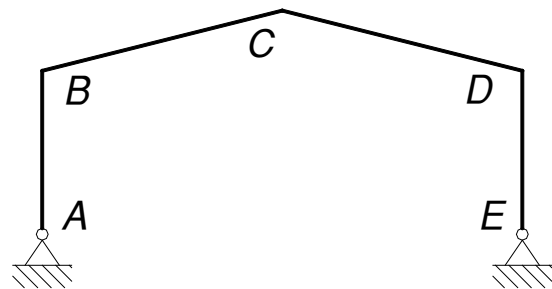
Semester 1, 2008/09**FIG. Q1(a)(i)****FIG. Q1(a)(ii)****FIG. Q1(b)(i)****FIG. Q1(b)(ii)**

Semester 1, 2009/10

**FIG. Q1(a)(i)****FIG. Q1(a)(ii)****FIG. Q1(b)(i)****FIG. Q1(b)(ii)**

Semester 1, 2010/11

**FIG. Q1(a)(i)****FIG. Q1(a)(ii)****FIG. Q1(b)(i)****FIG. Q1(b)(ii)**

Semester 1, 2011/12**FIG. Q1(a)(i)****FIG. Q1(a)(ii)****FIG. Q1(b)(i)****FIG. Q1(b)(ii)**